

AUTO-EQUIVALENCES OF STABLE MODULE CATEGORIES

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ABSTRACT. We construct nontrivial auto-equivalences of stable module categories for elementary, local symmetric algebras over a field k . These auto-equivalences are modeled after the spherical twists of Seidel and Thomas and the \mathbb{P}^n -twists of Huybrechts and Thomas, which yield auto-equivalences of the derived category of coherent sheaves on a variety. For group algebras of p -groups in characteristic p we recover many of the auto-equivalences corresponding to endo-trivial modules. We also obtain analogous auto-equivalences for local algebras of dihedral and semi-dihedral type, which are not group algebras.

1. INTRODUCTION

The goal of this article is to construct nontrivial auto-equivalences of the stable module category $\underline{\text{mod}}\text{-}A$ of a local symmetric algebra A over a field k , which are not induced by auto-equivalences of the derived category of A . The inspiration behind our constructions lies in the theory of endo-trivial modules for group algebras. In particular, suppose that k is a field of characteristic $p > 0$ and G a finite group of order divisible by p . A kG -module M is said to be *endo-trivial* if $\text{End}_k(M) \cong k \oplus P$ as kG -modules, where k is the trivial kG -module and P is projective. In this case the adjoint pair of exact functors $- \otimes_k M$ and $- \otimes_k M^*$ induces quasi-inverse auto-equivalences of $\underline{\text{mod}}\text{-}kG$. Of course, among these auto-equivalences are all (co)syzygy functors Ω^n for $n \in \mathbb{Z}$, corresponding to the endo-trivial modules $\Omega^n(k)$. Moreover, each syzygy functor Ω^n can naturally be lifted to an auto-equivalence of the derived category $D^b(\text{mod}\text{-}kG)$, corresponding to the tilting complex $A[-n]$. When G is a p -group, kG is a local algebra and hence all tilting complexes have the form $A[-n]$ for some $n \in \mathbb{Z}$. It follows that any auto-equivalence of the derived category $D^b(\text{mod}\text{-}kG)$ must act on objects as some power of the suspension functor composed with a Morita auto-equivalence, and thus the induced auto-equivalences of $\underline{\text{mod}}\text{-}kG$ are simply powers of Ω composed with Morita auto-equivalences. Therefore, when G admits an endo-trivial module M that is not a (co)syzygy of the trivial module, the stable equivalence induced by $- \otimes_k M$ is not induced by any derived auto-equivalence of kG . Such endotriivial modules arise for generalized dihedral, semidihedral and quaternion groups, and more generally for p -groups with a maximal elementary abelian subgroup of rank 2 [8].

The question remains of how one might generalize the definition of an endo-trivial module to arbitrary symmetric k -algebras. When G is a p -group, Carlson showed that M is endo-trivial if and only if $\underline{\text{End}}_{kG}(M) \cong k$ [5]. This condition generalizes easily enough to the case of a (local) symmetric k -algebra A : we may still regard an A -module M with $\underline{\text{End}}_A(M) \cong k$ as being ‘endo-trivial’. It is then clear that the image of the unique simple A -module under any stable auto-equivalence of A must be endo-trivial in this sense. However, absent the Hopf-algebra structure of a group algebra, there is no clear way to start with an endo-trivial A -module M and define an auto-equivalence of $\underline{\text{mod}}\text{-}A$ that sends the simple A -module to M^1 .

To get around this problem, we focus instead on the functors $- \otimes_k M$ for an endo-trivial kG -module M . Alperin’s construction of endo-trivial kG -modules as relative syzygies [1] provides us a guide for how the analogous functors on $\underline{\text{mod}}\text{-}A$ should behave on various subcategories of $\underline{\text{mod}}\text{-}A$ and how such functors can be defined via bimodules related to an appropriate subalgebra of A . However, the actual constructions we obtain, and elements of the proofs, are closely modeled on certain Fourier Mukai transforms that give equivalences between derived categories of coherent sheaves. Our first construction is similar to a 0-spherical twist introduced by Seidel and Thomas [20], while our second is similar to the \mathbb{P}^n -twists defined by Huybrechts and Thomas [13]. We note that Joseph Grant has already adapted spherical twists and \mathbb{P}^n -twists to construct auto-equivalences of the derived category of a symmetric algebra, which he called *periodic twists* [11]. While

¹This problem can be viewed as a special case of the more general question: if S is a simple-minded system [14, 9] in $\underline{\text{mod}}\text{-}A$ is there necessarily an algebra B and an equivalence $\alpha : \underline{\text{mod}}\text{-}B \rightarrow \underline{\text{mod}}\text{-}A$ sending the simple B -modules to S ?

there are ample similarities, we do not see any obvious way in which these constructions could be directly related.

The paper is organized as follows. We start by reviewing some results of Bridgeland that give sufficient conditions for an exact functor between triangulated categories to be fully faithful or an equivalence. These results were originally applied in the setting of derived categories of coherent sheaves on a projective variety, and Grant has recently applied them in the setting of the derived category of a symmetric algebra [11]. As our focus is on stable module categories, we prove a version of Bridgeland's results with respect to a stronger notion of a spanning class (see Proposition 2.2(2)). In the next section we focus on stable module categories for symmetric algebras, and the exact functors between such categories that are induced by tensoring with so-called *left-right projective* bimodules. In Theorem 3.4, we adapt Bridgeland's ideas to give a necessary and sufficient condition for such a functor to induce a stable equivalence in terms of the images of the simple modules and their extensions. We note that the equivalences that arise in this way are the stable equivalences of *Morita type*. In Section 4, we focus on the situation in which we have an elementary, local symmetric k -algebra A that is free as a module on both sides over a subalgebra R isomorphic to $k[x]/(x^m)$. We prove some preliminary results under the assumption that the induced module $k \otimes_R A$ has a stable endomorphism algebra isomorphic to $k[\psi]/(\psi^{n+1})$. The case where $n = 1$ is considered first in Section 5. Here, we define a *spherical stable twist* and show that it is a non-trivial auto-equivalence of $\underline{\text{mod}}\text{-}A$. In Section 6, we define \mathbb{P}^n *stable twists* (for $n \geq 1$) and show that they also yield auto-equivalences of $\underline{\text{mod}}\text{-}A$. Our definition makes use of a double cone construction that is very similar to that used by Huybrechts and Thomas in their construction of \mathbb{P}^n twists in [13]. It would be interesting to know if there is a common generalization of these two constructions.

Finally, Section 7 returns to some of the examples of endo-trivial modules for p -groups. We show not only that these endo-trivial modules are obtained as the images of the trivial module under a spherical or \mathbb{P}^n stable twist, but also that these auto-equivalences continue to be defined for local algebras of dihedral or semidihedral type, which are not group algebras. We note that our auto-equivalences do not appear to recover the endo-trivial modules for generalized quaternion 2-groups. Based on Theorem 3.1(d) of [8], it seems plausible that there is also a quadruple cone construction that can be used to define auto-equivalences of the stable category, and that will reproduce the endo-trivial modules for the generalized quaternion 2-groups. Moreover the existence of such a construction would suggest, along the lines of [11], that it may even be possible to construct auto-equivalences of $\underline{\text{mod}}\text{-}A$ starting from a module whose stable endomorphism algebra is periodic.

2. EQUIVALENCES OF TRIANGULATED CATEGORIES

We begin by reviewing some general results that are useful in establishing that a given exact functor between triangulated categories is an equivalence. Let k be a field and let \mathcal{T} be a Hom-finite triangulated k -category with suspension Σ . For objects $X, Y \in \mathcal{T}$ we write $\mathcal{T}(X, Y)$ for the morphisms from X to Y , and we occasionally write $X[n]$ for $\Sigma^n X$. Recall that an exact auto-equivalence ν of \mathcal{T} is said to be a Serre functor if there are natural isomorphisms $\mathcal{T}(X, Y) \cong D\mathcal{T}(Y, \nu X)$ for all $X, Y \in \mathcal{T}$.

Definition 2.1. A collection \mathcal{C} of objects in a triangulated category \mathcal{T} is a **spanning class** of \mathcal{T} if

- $\mathcal{T}(X, Y[i]) = 0$ for all $X \in \mathcal{C}$ and all $i \in \mathbb{Z}$ implies $Y \cong 0$; and
- $\mathcal{T}(Y[i], X) = 0$ for all $X \in \mathcal{C}$ and all $i \in \mathbb{Z}$ implies $Y \cong 0$.

A collection \mathcal{D} of objects in \mathcal{T} will be said to be a **strong spanning class** if

- $\mathcal{T}(X, Y) = 0$ for all $X \in \mathcal{D}$ implies $Y \cong 0$; and
- $\mathcal{T}(Y, X) = 0$ for all $X \in \mathcal{D}$ implies $Y \cong 0$.

Observe that if \mathcal{T} has a Serre functor, then the two conditions in either of the above definitions are equivalent. Thus it suffices to check only one of the conditions in this case. In particular, when \mathcal{T} has a Serre functor, useful examples of (strong) spanning classes are of the form $\mathcal{C} = \{X\} \cup X^\perp$ for any $X \in \mathcal{T}$, where $X^\perp = \{Y \in \mathcal{T} \mid \mathcal{T}(X, Y) = 0\}$. Another example of a strong spanning class is a maximal system of orthogonal bricks (see [9]).

Now suppose that \mathcal{T} and \mathcal{T}' are triangulated categories with suspensions Σ and Σ' respectively. If $F : \mathcal{T} \rightarrow \mathcal{T}'$ is an exact equivalence of triangulated categories, then clearly it has a left and right adjoint, and

it can be seen that this adjoint is also exact. More generally, suppose that $(F, \sigma) : \mathcal{T} \rightarrow \mathcal{T}'$ is an exact functor, where $\sigma : F\Sigma \xrightarrow{\cong} \Sigma'F$ is an isomorphism, and suppose that $H : \mathcal{T}' \rightarrow \mathcal{T}$ is right adjoint to F . Notice that this means that H is a k -linear functor and there is a natural isomorphism $\eta : \mathcal{T}'(F(-), -) \xrightarrow{\cong} \mathcal{T}(-, H(-))$. We can define an isomorphism $\tau : H\Sigma' \rightarrow \Sigma H$ via the sequence of isomorphisms of bi-functors

$$(2.1) \quad \begin{array}{ccccc} \mathcal{T}(\Sigma(-), H\Sigma'(-)) & \xrightarrow{\eta^{-1}} & \mathcal{T}'(F\Sigma(-), \Sigma'(-)) & \xrightarrow{\mathcal{T}'(\sigma, \Sigma'(-))^{-1}} & \mathcal{T}'(\Sigma'F(-), \Sigma'(-)) \\ \downarrow \tau(\Sigma(-), \tau) & & & & \downarrow \Sigma'^{-1} \\ \mathcal{T}(\Sigma(-), \Sigma H(-)) & \xleftarrow{\Sigma} & \mathcal{T}(-, H(-)) & \xleftarrow{\eta} & \mathcal{T}'(F(-), -). \end{array}$$

We shall write $h : 1 \rightarrow HF$ for the unit of the adjunction, so that $h_Y = \eta(1_{FY})$ for any object Y of \mathcal{T} . Starting with 1_{FY} in the bottom right of the above diagram, we see that

$$(2.2) \quad \Sigma(h_Y) = \Sigma(\eta(1_{FY})) = \tau_{FY} \circ \eta(\Sigma'(1_{FY})\sigma_Y) = \tau_{FY} \circ \eta(\sigma_Y) = \tau_{FY} \circ H(\sigma_Y) \circ h_{\Sigma Y}.$$

This equality expresses the fact that the unit of the adjunction is compatible with the suspension functor in the appropriate sense. Dually, for the co-unit $g : FH \rightarrow 1$, one obtains $\Sigma'(g_X) \circ \sigma_{HX} \circ F(\tau_X) = g_{\Sigma'X}$ for any object X of \mathcal{T}' .

Proposition 2.2 (Cf. [4]). *Let $F : \mathcal{T} \rightarrow \mathcal{T}'$ be an exact functor between triangulated categories with a right adjoint H and a left adjoint G . Then F is fully faithful if either of the following two conditions is satisfied.*

- (1) *There exists a spanning class \mathcal{C} of \mathcal{T} such that the natural homomorphisms*

$$F : \mathcal{T}(X, Y[i]) \rightarrow \mathcal{T}'(FX, F(Y[i]))$$

are bijective for all $X, Y \in \mathcal{C}$ and all $i \in \mathbb{Z}$.

- (2) *There exists a strong spanning class \mathcal{D} of \mathcal{T} such that for all $X, Y \in \mathcal{D}$ the natural homomorphisms*

$$F : \mathcal{T}(X, Y[i]) \rightarrow \mathcal{T}'(FX, F(Y[i]))$$

are bijective for $i = 0$ and injective for $i = 1$.

Proof. The first condition corresponds to a theorem of Bridgeland in [4]. We prove the sufficiency of the second condition in a similar manner. Consider the commutative diagram

$$(2.3) \quad \begin{array}{ccc} \mathcal{T}(X, Y) & \xrightarrow{h_Y \circ} & \mathcal{T}(X, HFY) \\ \circ g_X \downarrow & \searrow F & \downarrow \cong \\ \mathcal{T}(GFX, Y) & \xrightarrow{\cong} & \mathcal{T}'(FX, FY) \end{array}$$

where g denotes the co-unit of the adjunction $G \dashv F$ and h denotes the unit of the adjunction $F \dashv H$. Let $X \in \mathcal{D}$ and construct a triangle $GFX \xrightarrow{g_X} X \rightarrow Z \rightarrow$. If we apply $\mathcal{T}(-, Y)$ to this triangle we get an exact sequence

$$\begin{array}{ccccccc} \mathcal{T}(X, Y) & \xrightarrow{\circ g_X} & \mathcal{T}(GFX, Y) & \longrightarrow & \mathcal{T}(Z[-1], Y) & \longrightarrow & \mathcal{T}(X[-1], Y) \xrightarrow{\circ \Sigma^{-1}(g_X)} \mathcal{T}(GFX[-1], Y) \longrightarrow \\ & \searrow F & \downarrow \cong & & \downarrow \Sigma \cong & & \downarrow \Sigma \cong \\ & & \mathcal{T}'(FX, FY) & & \mathcal{T}(X, Y[1]) & \xrightarrow{\circ g_X} & \mathcal{T}(GFX, Y[1]) \\ & & & & \searrow F & & \downarrow \cong \\ & & & & & & \mathcal{T}'(FX, F(Y[1])) \end{array}$$

If $Y \in \mathcal{D}$, the first map labeled F is bijective and the second is injective. Thus the commutativity of the diagram yields $\mathcal{T}(Z[-1], Y) = 0$. Consequently, $Z \cong 0$ and g_X is an isomorphism. It follows from the commutativity of (2.3) that for any $X \in \mathcal{D}$, $h_Y \circ : \mathcal{T}(X, Y) \rightarrow \mathcal{T}(X, HFY)$ is bijective for all $Y \in \mathcal{T}$. We now apply $\mathcal{T}(X, -)$ with $X \in \mathcal{D}$ to the triangle $Y \xrightarrow{h_Y} HFY \rightarrow W \rightarrow$ to get $\mathcal{T}(X, W) = 0$. Here we are

using the fact that $\mathcal{T}(X, \Sigma(h_Y))$ is bijective, which follows from (2.2) since we know $\mathcal{T}(X, h_{\Sigma Y})$ is bijective. Hence $W \cong 0$ and h_Y is an isomorphism for all $Y \in \mathcal{T}$. Returning to the commutative square (2.3) we see that F must induce an isomorphism for all $X, Y \in \mathcal{T}$. \square

Once we have established that a given functor F is fully faithful, we can use the following result to verify that F is an equivalence of triangulated categories.

Proposition 2.3 (Bridgeland [4]). *Let $F : \mathcal{T} \rightarrow \mathcal{T}'$ be a fully faithful exact functor between triangulated categories such that \mathcal{T} contains nonzero objects and \mathcal{T}' is indecomposable. Then F is an equivalence of categories if and only if F has a left adjoint G and a right adjoint H such that $H(Y) \cong 0$ implies $G(Y) \cong 0$ for any $Y \in \mathcal{T}'$.*

Notice, in particular, that if G is both a left and right adjoint to F in the above situation, then F is an equivalence if and only if it is fully faithful.

3. LEFT-RIGHT PROJECTIVE BIMODULES

Let A and B be basic, indecomposable and non-semisimple self-injective k -algebras, where k is a field. We consider (A, B) -bimodules on which k acts centrally, which we identify with right modules over $A^{\text{op}} \otimes_k B$. If $A = B$, this ring is also called the enveloping algebra of A and is denoted A^e . We let $D = \text{Hom}_k(-, k)$ be the duality with respect to the ground field. For an (A, B) -bimodule M , we get a (B, A) -bimodule DM . We may also take the dual of M with respect to either A or B on one side. We set ${}^*M = \text{Hom}_A(M, A)$ and $M^* = \text{Hom}_B(M, B)$.

We are interested in functors of the form $- \otimes_A M_B : \text{mod-}A \rightarrow \text{mod-}B$ for a bimodule ${}_A M_B$. This functor is exact if and only if ${}_A M$ is projective, and it takes projective A -modules to projective B -modules if and only if M_B is projective. We thus call the bimodule ${}_A M_B$ **left-right projective** if ${}_A M$ and M_B are both projective. For such bimodules the functor $- \otimes_A M_B$ induces an exact functor between triangulated categories $\underline{\text{mod-}}A \rightarrow \underline{\text{mod-}}B$.

Observe that $\Lambda := A^{\text{op}} \otimes_k B$ is self-injective, and thus $\text{mod-}\Lambda$ is a Frobenius category. The subcategory $\text{lrp}(A, B)$ of left-right projective bimodules is an exact subcategory containing the projective bimodules, and hence it is also a Frobenius category. We write $\underline{\text{lrp}}(A, B)$ for the corresponding stable category of left-right projective bimodules, and note that it inherits the structure of a triangulated category. A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle if and only if there is a short exact sequence $0 \rightarrow X \xrightarrow{\begin{pmatrix} f \\ p \end{pmatrix}} Y \oplus P \xrightarrow{(g \ q)} Z \rightarrow 0$ corresponding to the map $\underline{h} \in \text{Ext}_{\Lambda}^1(Z, X) \cong \underline{\text{Hom}}_{\Lambda}(Z, \Omega^{-1}X)$.

Lemma 3.1. *Let M be a left-right projective (A, B) -bimodule and let $\Lambda = A^{\text{op}} \otimes_k B$. Then we have isomorphisms in $\underline{\text{lrp}}(A, B)$*

$$\Omega_{\Lambda}(M) \cong M \otimes_B \Omega_{B^e}(B) \cong \Omega_{A^e}(A) \otimes_A M.$$

Proof. We have an exact sequence $0 \rightarrow \Omega_{A^e}(A) \rightarrow P \rightarrow A \rightarrow 0$ in $\text{lrp}(A, A)$ for a projective A^e -module P . If we tensor on the right with M , we obtain an exact sequence $0 \rightarrow \Omega_{A^e}(A) \otimes_A M \rightarrow P \otimes_A M \rightarrow M \rightarrow 0$ in $\text{lrp}(A, B)$. Since M_B is projective, $P \otimes_A M$ is a projective (A, B) -bimodule, and it follows that $\Omega_{A^e}(A) \otimes_A M \cong \Omega_{\Lambda}(M)$ up to projective summands. The other isomorphism is proved similarly. \square

Let M be a left-right projective (A, B) -bimodule. Since A and B are self-injective DM, M^* and *M are all left-right projective (B, A) -modules. The functor $- \otimes_A M_B$ has a right adjoint $\text{Hom}_B(M, -) \cong - \otimes_B M^*$ and a left adjoint $- \otimes_A {}^*M$. Furthermore, it is not hard to see that these functors induce adjoint pairs of functors between $\underline{\text{mod-}}A$ and $\underline{\text{mod-}}B$.

Lemma 3.2. *For an (A, B) -bimodule M , we have isomorphisms $DA \otimes_A M \cong D({}^*M)$ and $M \otimes_B DB \cong D(M^*)$ of (A, B) -bimodules. Hence $M^* \cong {}^*M$ if and only if $DA \otimes_A M \cong M \otimes_B DB$ as bimodules. In particular, if A and B are symmetric algebras, then $M^* \cong {}^*M \cong DM$.*

Proof. Using Hom-tensor adjunction, we have natural isomorphisms

$$\begin{aligned} D(DA \otimes_A M) &= \text{Hom}_k(DA \otimes_A M, k) \cong \text{Hom}_A({}_A M, \text{Hom}_k(DA, k)) \\ &\cong \text{Hom}_A({}_A M, A) = {}^*M. \end{aligned}$$

By naturality, these are bimodule isomorphisms. The isomorphism $D(M^*) \cong M \otimes_B DB$ is proved similarly. Finally, if A and B are symmetric, then $DA \cong A$ and $DB \cong B$ as bimodules. Thus $M \cong D(M^*) \cong D({}^*M)$, and applying the duality D gives the desired result. \square

We have the following consequence of Proposition 2.3. The assumptions on B guarantee that $\underline{\text{mod}}\text{-}B$ is indecomposable.

Corollary 3.3. *Let ${}_A M_B$ be a left-right projective bimodule and assume A and B are symmetric algebras with B indecomposable and of Loewy length greater than 2. Then $- \otimes_A M_B : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}B$ is an equivalence if and only if it is fully faithful.*

Using the obvious fact that the set of simple A -modules is a strong spanning class in $\underline{\text{mod}}\text{-}A$, we can apply Proposition 2.2(2) and Bridgeland's results from the last section to give a characterization of the left-right projective bimodules that induce stable equivalences. This result might be seen as a rough analogue of Bridgeland's Theorem 1.1 in [4] describing which vector bundles induce Fourier-Mukai transforms. For simplicity, we assume that A and B are split over the field k . In particular, every simple module has endomorphism ring isomorphic to k .

Theorem 3.4. *Suppose A and B are split, indecomposable symmetric k -algebras with B of Loewy length greater than 2. Let M be a left-right projective (A, B) -bimodule and let $F : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}B$ be the functor induced by $- \otimes_A M_B$. Then F is an equivalence if and only if*

$$\underline{\text{Hom}}_B(F(S_i), F(S_j)) \cong \begin{cases} k, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

and F induces monomorphisms

$$\text{Ext}_A^1(S_i, S_j) \rightarrow \text{Ext}_B^1(F(S_i), F(S_j))$$

for all i, j .

Under some mild assumption on the ground field or the semisimple quotients of the algebras A and B , any equivalence of stable categories induced by a left-right projective bimodule is of Morita type. Recall that a pair of bimodules ${}_A M_B$ and ${}_B N_A$ is said to induce a *stable equivalence of Morita type* between A and B if we have isomorphisms of (A, A) and (B, B) -bimodules, respectively:

$$M \otimes_B N \cong A \oplus P \quad \text{and} \quad N \otimes_A M \cong B \oplus Q$$

for projective bimodules P and Q . A (semisimple) k -algebra R is said to be *separable* if $R \otimes_k K$ is semisimple for any field K containing k . This holds automatically if k is perfect, or if R splits over k . In particular, the following theorem applies to the elementary, local symmetric k -algebras that we study in the following sections.

Theorem 3.5 (Rickard [18]). *Assume $A/\text{rad}A$ and $B/\text{rad}B$ are separable k -algebras, and ${}_A M_B$ is an indecomposable, left-right projective bimodule such that $- \otimes_A M_B$ induces an equivalence $\underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}B$. Then M and $N := M^*$ give a stable equivalence of Morita type between A and B .*

The following result of Linckelmann provides a useful way of comparing different stable equivalences of Morita type.

Theorem 3.6 (Prop. 2.5 in [15]). *Suppose that A and B are self-injective k -algebras with no projective simple modules, and ${}_A M_B$ is a left-right projective bimodule for which $- \otimes_A M_B$ induces a stable equivalence $\underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}B$. Then $- \otimes_A M_B : \text{mod}\text{-}A \rightarrow \text{mod}\text{-}B$ induces a Morita equivalence if and only if $S \otimes_A M_B$ is simple for each simple A -module S .*

4. SET UP AND PRELIMINARY RESULTS

We now focus on an elementary, local, symmetric k -algebra A . Recall that A elementary means that $A/\text{rad}A \cong \prod_{i=1}^s k$ as k -algebras for some $s \geq 1$, and we must have $s = 1$ since A is local. Furthermore, it follows that A is a split k -algebra with a unique simple (right) module $k \cong A/\text{rad}A$, up to isomorphism. Let $x \in \text{rad}A$ be an element with $x^m = 0$ but $x^{m-1} \neq 0$ for some $m \geq 2$, and set $R = k \cdot 1 + k \cdot x + \cdots + k \cdot x^{m-1} \cong k[t]/(t^m)$, the unital subalgebra of A generated by x . We shall assume that A is free as an R -module on both sides. Thus we have exact induction and restriction functors $- \otimes_R A : \text{mod-}R \rightarrow \text{mod-}A$ and $- \otimes_A A_R : \text{mod-}A \rightarrow \text{mod-}R$. Since R and A are symmetric algebras, and ${}_A A_R$ and ${}_R A_A$ are left-right projective bimodules, induction and restriction are left and right adjoints of each other. Furthermore, they induce mutually adjoint exact functors between $\underline{\text{mod-}}R$ and $\underline{\text{mod-}}A$. When convenient we will write $F = - \otimes_R A$ for induction and $G = - \otimes_A A_R$ for restriction. We also let $\eta : \underline{\text{Hom}}_A(FX, Y) \xrightarrow{\cong} \underline{\text{Hom}}_R(X, GY)$ and $\theta : \underline{\text{Hom}}_R(GX, Y) \xrightarrow{\cong} \underline{\text{Hom}}_A(X, FY)$ denote the adjugants.

We also write k for the unique simple (right) R -module R/xR , and we set $T_A := k \otimes_R A \cong A/xA$. Notice that $\text{End}_A(T) \cong \{a \in A \mid ax \in xA\}/xA$ and $\underline{\text{End}}_A(T) \cong \{a \in A \mid ax \in xA\}/(xA + \text{ann}_l(x))$. Our key assumption in constructing auto-equivalences of $\underline{\text{mod-}}A$ is that

$$\underline{\text{End}}_A(T) \cong k[\psi]/(\psi^{n+1})$$

for some $n \geq 1$. By the above description of $\underline{\text{End}}_A(T)$, we may assume that ψ is induced by left multiplication by an element $y \in \text{rad}A$, which we henceforth fix.

Lemma 4.1. *Assume that $\underline{\text{End}}_A(T) = k[\psi]/(\psi^{n+1})$ with ψ induced by left multiplication by an element $y \in A$. Then $T_R \cong k^{n+1} \oplus R^l$ for some integer $l \geq 0$ and the restriction of ψ coincides with the following $(n+1) \times (n+1)$ matrix (with respect to an appropriate basis)*

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \underline{\text{End}}_R(T).$$

Proof. We first decompose $T_R = U_R \oplus R^l$ where U_R is the sum of all nonprojective direct summands of T_R . Since induction is left adjoint to restriction we have an isomorphism $\eta : \underline{\text{Hom}}_A(T, T) \xrightarrow{\cong} \underline{\text{Hom}}_R(k, T_R) = \underline{\text{Hom}}_R(k, U) = \text{Hom}_R(k, U)$. Thus $\text{soc } U$ has dimension $n+1$, and U must decompose as a direct sum of $n+1$ indecomposable R -modules (which are all uniserial). Furthermore, $\underline{\text{Hom}}_R(k, T_R) = \text{Hom}_R(k, U)$ is spanned by the set

$$\{\eta(1_T), \eta(\psi), \dots, \eta(\psi^n)\} = \{\eta(1_T), y\eta(1_T), \dots, y^n\eta(1_T)\}.$$

Consequently $\text{soc } U$ is spanned by $\{y^i\eta(1_T)(1)\}_{0 \leq i \leq n}$. Observe that $\eta(1_T)$ corresponds to the map $k \rightarrow A/xA$ sending $1 \in k$ to $1 + xA \in A/xA$, which is a split monomorphism. Furthermore, if $f : A/xA \rightarrow k$ is a splitting for $\eta(1_T)$, then $f \circ \psi = 0$ as the map ψ corresponds to multiplication by an element $y \in \text{rad}A$ on which f vanishes.

Since induction is also right adjoint to restriction, we have an isomorphism $\theta : \underline{\text{Hom}}_A(T, T) \xrightarrow{\cong} \underline{\text{Hom}}_R(T_R, k) = \underline{\text{Hom}}_R(U, k) = \text{Hom}_R(U, k)$. Hence $\text{Hom}_R(U, k)$ is spanned by

$$\{\theta(1_T), \theta(\psi), \dots, \theta(\psi^n)\} = \{\theta(1_T), \theta(1_T)y, \dots, \theta(1_T)y^n\}.$$

As the map $f : T \rightarrow k$ above has $fy = 0$, we must have $f = \theta(1_T)y^n$ up to a scalar multiple. As $f(\eta(1_T)(1)) = 1$, we have $\theta(1_T)y^{n-i}(y^i\eta(1_T)(1)) = 1$ for each i . Consequently, the elements of the socle of U are not in $\text{rad } U$ and it follows that they generate U . Therefore $U = \text{soc } U \cong k^{n+1}$, and y has the desired matrix with respect to the basis $\{y^i\eta(1_T)(1)\}_{0 \leq i \leq n}$. \square

Remark. The converse of the above lemma also appears to be true. However, when $n = 1$, it is enough to verify that $T_R \cong k^2 \oplus R^l$. For then $\underline{\text{End}}_A(T)$ is a two-dimensional local ring and hence must be isomorphic to $k[\psi]/(\psi^2)$.

When $m = 2$, the short exact sequence $0 \rightarrow k \rightarrow R \rightarrow k \rightarrow 0$ yields an exact sequence of A -modules $0 \rightarrow T \rightarrow A \rightarrow T \rightarrow 0$ which shows that $\Omega T \cong T$. For $m > 2$, the projective presentation $0 \rightarrow k \rightarrow R \xrightarrow{x} R \rightarrow k \rightarrow 0$ yields a projective presentation for T of the form $0 \rightarrow T \rightarrow A \xrightarrow{x} A \rightarrow T \rightarrow 0$, showing that $\Omega^2 T \cong T$. In this case we will need some additional information about the maps between ΩT and T .

Lemma 4.2. *Let R, A and $T = k \otimes_R A$ be as above. Then $\underline{\text{Hom}}_A(\Omega T, T)$ has a k -basis $\{\xi, \psi\xi, \dots, \psi^n\xi\}$ where $\xi = f \otimes_R A$ for a nonzero map $f : \Omega k_R \rightarrow k_R$. Furthermore, provided that either $n = 1$ or that x^{m-2} commutes with y , the restriction of ξ is given by the diagonal matrix*

$$\begin{pmatrix} f & & 0 \\ & \ddots & \\ 0 & & f \end{pmatrix}$$

with respect to appropriate bases of the projective-free summands of ΩT_R and T_R ,

Proof. We have an isomorphism $\eta : \underline{\text{Hom}}_A(\Omega T, T) \xrightarrow{\cong} \underline{\text{Hom}}_R(\Omega k_R, T_R)$. Using the same basis for the projective-free part of $T_R \cong k^{n+1}$ as in the last lemma, we have

$$\eta(\xi) = \eta(F(f)) = \eta(1_T) \circ f = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} f = \begin{pmatrix} f \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and

$$\eta(\psi^i \xi) = G(\psi^i) \eta(\xi) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^i \begin{pmatrix} f \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ f \\ \vdots \\ 0 \end{pmatrix}$$

with f in the $(i+1)^{th}$ row. Thus η sends $\{\psi^i \xi\}_{0 \leq i \leq n}$ to a basis for $\underline{\text{Hom}}_R(\Omega k_R, T_R)$ and the first conclusion follows.

For the second claim, observe that $(\Omega T)_R \cong \Omega(T_R) \cong \Omega(k_R^{n+1}) \cong (\Omega k_R)^{n+1}$ in $\underline{\text{mod}}\text{-}R$. As in the proof of Lemma 4.1, we fix such a decomposition of ΩT_R so that $\eta(\Omega(\psi^i))$ is the identity map from Ωk_R to the $(i+1)^{th}$ component of this direct sum decomposition. Then

$$\eta(\xi \circ \Omega(\psi^i)) = G(\xi) \circ \eta(\Omega(\psi^i)) = G(\xi) \circ \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus with respect to the chosen decompositions, $G(\xi) : \Omega k_R^{n+1} \rightarrow k_R^{n+1}$ corresponds to the matrix with $\eta(\xi \circ \Omega(\psi^i))$ in the $(i+1)^{th}$ column. Since $\psi^i : x^{m-1}A \rightarrow x^{m-1}A$ and $\Omega(\psi^i) : xA \rightarrow xA$ are both induced by left multiplication by y^i , while $\xi : xA \rightarrow x^{m-1}A$ is induced by multiplication by x^{m-2} , if x^{m-2} commutes with y we obtain $\xi \circ \Omega(\psi^i) = \psi^i \xi$ for each i . Hence the $(i+1)^{th}$ column of $G(\xi)$ coincides with

$$\eta(\psi^i \xi) = \begin{pmatrix} 0 \\ \vdots \\ f \\ \vdots \\ 0 \end{pmatrix}$$

with f in the $(i+1)^{th}$ row as computed above.

Alternatively, if $n = 1$ we can write $\xi\Omega(\psi) = a\xi + b\psi\xi \in \underline{\text{Hom}}_A(\Omega T, T)$ for $a, b \in k$ with $b \neq 0$. If $a \neq 0$ we would obtain $\psi\xi = a^{-1}\psi\xi\Omega(\psi)$, and thus $\psi\xi = a^{-2}\psi\xi\Omega(\psi)^2 = 0$, which is a contradiction. Thus $\xi\Omega(\psi)$ equals a scalar multiple of $\psi\xi$. As above, it follows that the second column of $G(\xi)$ would be $\begin{pmatrix} 0 \\ b_f \end{pmatrix}$, and scaling the second generator of $\Omega(T_R)$ by b^{-1} allows us to remove the b . \square

5. SPHERICAL STABLE TWISTS

The stable equivalences we construct in this section appear quite similar to the 0-spherical twists introduced by Seidel and Thomas [20], and we will see later that they arise naturally for group algebras of certain 2-groups.

Definition 5.1. *Let A be an elementary local symmetric k -algebra, and let R be a unital subalgebra of A such that ${}_R A$ and A_R are free. Let $\mu : A \otimes_R A \rightarrow A$ be induced by multiplication and write C_μ for the cone of μ in the category $\underline{\text{lrp}}(A, A)$. We write $\tau_R : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$ for the endo-functor induced by $-\otimes_A C_\mu$.*

If $R = k[x] \subset A$ for an element $x \in A$ with $x^m = 0$, and $k \otimes_R A_R \cong k^2 \oplus R^l$, then we will call τ_R a spherical stable twist.

Remark. In case $m = 2$, $T = k \otimes_R A$ satisfies $T[1] \cong T$ and $\underline{\text{End}}_A(T) \cong k[\psi]/(\psi^2)$. Thus T resembles a 0-spherical object as defined by Seidel and Thomas [20]. Likewise we regard the functor τ_R as an analog of a 0-spherical twist in this case.

Theorem 5.2. *A spherical stable twist τ_R is an exact auto-equivalence of $\underline{\text{mod}}\text{-}A$.*

Proof. Since the left and right adjoints of τ_R coincide, it suffices to show that τ_R is fully faithful by Proposition 2.3. Observe that $T[2] \cong T$ and $\mathcal{C} = \{T\} \cup T^\perp$ is a spanning class for $\underline{\text{mod}}\text{-}A$. Furthermore $T^\perp = \{X \in \underline{\text{mod}}\text{-}A \mid \underline{\text{Hom}}_A(T, X) = 0\} = \{X \in \underline{\text{mod}}\text{-}A \mid \underline{\text{Hom}}_R(k, X_R) = 0\} = \{X \in \underline{\text{mod}}\text{-}A \mid X_R \text{ is projective}\}$. Since A_R is projective, T^\perp is closed under (de-)suspensions. Thus, according to Proposition 2.2, to show that τ_R is fully faithful, it suffices to check that it induces bijections $\underline{\text{Hom}}_A(T, T) \rightarrow \underline{\text{Hom}}_A(\tau_R(T), \tau_R(T))$, $\underline{\text{Hom}}_A(\Omega T, T) \rightarrow \underline{\text{Hom}}_A(\tau_R(\Omega(T)), \tau_R(T))$, and $\underline{\text{Hom}}_A(X, Y) \rightarrow \underline{\text{Hom}}_A(\tau_R(X), \tau_R(Y))$ for all $X, Y \in T^\perp$.

If $X \in T^\perp$, then $\tau_R(X)$ is defined as the cone of the natural map $X \otimes_R A \rightarrow X$. Since X_R is projective, so is $X \otimes_R A$ and thus we have an isomorphism $X \rightarrow \tau_R(X)$ in $\underline{\text{mod}}\text{-}A$, and it is clearly natural with respect to $X \in T^\perp$. Thus τ_R is fully faithful on T^\perp .

We now consider $\tau_R(T)$. For ease of notation, we let F and G be the functors induced by induction and restriction respectively between $\underline{\text{mod}}\text{-}R$ and $\underline{\text{mod}}\text{-}A$. We let $\eta : \underline{\text{Hom}}_A(FX, Y) \xrightarrow{\cong} \underline{\text{Hom}}_R(X, GY)$ denote the adjugant. Observe that for any $X \in \underline{\text{mod}}\text{-}A$, the counit $\delta_X = \eta^{-1}(1_{GX}) : FGX \rightarrow X$ coincides with the natural map $X \otimes_R A \rightarrow X$ induced by tensoring the map μ with X_A on the left. By Lemma 4.1 $GT \cong k^2$ and with respect to the basis given in the proof, we have $\eta(1) : k \rightarrow GT$ corresponding to the map $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\eta(\psi) : k \rightarrow GT$ corresponding to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Hence $F(\eta(1)) = \begin{pmatrix} 1_T \\ 0 \end{pmatrix} : T \rightarrow FGT \cong T^2$ and $F(\eta(\psi)) = \begin{pmatrix} 0 \\ 1_T \end{pmatrix} : T \rightarrow FGT \cong T^2$. Now $\delta_T : FGT \cong T^2 \rightarrow T$ has the form $(u \ v)$ for $u, v \in \underline{\text{End}}_A(T)$. Since $1_T = \delta_T F(\eta(1_T)) = \delta_T \begin{pmatrix} 1_T \\ 0 \end{pmatrix} = u$ and $\psi = \delta_T F(\eta(\psi)) = \delta_T \begin{pmatrix} 0 \\ 1_T \end{pmatrix} = v$, we see that $\delta_T = (1_T, \psi)$ with respect to this decomposition $FGT \cong T^2$. Now consider the induced isomorphism of split triangles

$$(5.1) \quad \begin{array}{ccccc} \tau_R(T)[-1] & \longrightarrow & FG(T) & \longrightarrow & T \longrightarrow \\ \downarrow \cong & & \downarrow \cong & & \parallel \\ T & \xrightarrow{\begin{pmatrix} -\psi \\ 1 \end{pmatrix}} & T^2 & \xrightarrow{(1 \ \psi)} & T \longrightarrow \end{array}$$

which shows $\tau_R(T)[-1] \cong T$ and thus $\tau_R(T) \cong \Sigma(T)$. By Lemma 4.1 we have $FG(\psi) = \begin{pmatrix} 0 & 0 \\ 1_T & 0 \end{pmatrix} : FGT \rightarrow FGT$, and using the above isomorphism of triangles we compute $\tau_R(\psi) = \Sigma(-\psi)$:

$$\begin{array}{ccccccc}
T & \xrightarrow{\begin{pmatrix} -\psi \\ 1 \end{pmatrix}} & T^2 & \xrightarrow{(1 \ \psi)} & T & \xrightarrow{0} & \tau_R(T) \cong \Sigma(T) \\
\downarrow \scriptstyle -\psi & & \downarrow \scriptstyle \begin{pmatrix} 0 & 0 \\ 1_T & 0 \end{pmatrix} & & \downarrow \scriptstyle \psi & & \downarrow \scriptstyle \Sigma(-\psi) \\
T & \xrightarrow{\begin{pmatrix} -\psi \\ 1 \end{pmatrix}} & T^2 & \xrightarrow{(1 \ \psi)} & T & \xrightarrow{0} & \tau_R(T) \cong \Sigma(T)
\end{array}$$

Thus τ_R induces an isomorphism $\underline{\text{End}}_A(T) \cong \underline{\text{End}}_A(\tau_R(T))$ as required.

Next we consider the action of τ_R on the map $\xi \in \underline{\text{Hom}}_A(\Omega T, T)$ as in Lemma 4.2. As in the proof of that lemma, we have a nonzero scalar $b \in k$ such that $\xi\Omega(\psi) = b\psi\xi$. If we suspend the isomorphism of triangles (5.1), we can use it to compute $\tau_R(\xi) = \Sigma(b\xi)$ as in the diagram below, where the map $\Omega T^2 \rightarrow T^2$ is $FG(\xi)$ according to Lemma 4.2.

$$\begin{array}{ccccccc}
\Omega T & \xrightarrow{\begin{pmatrix} -\Omega(\psi) \\ 1 \end{pmatrix}} & \Omega T^2 & \xrightarrow{(1 \ \Omega(\psi))} & \Omega T & \xrightarrow{0} & \tau_R(\Omega(T)) \cong \Sigma(\Omega T) \\
\downarrow \scriptstyle b\xi & & \downarrow \scriptstyle \begin{pmatrix} \xi & 0 \\ 0 & b\xi \end{pmatrix} & & \downarrow \scriptstyle \xi & & \downarrow \scriptstyle \Sigma(b\xi) \\
T & \xrightarrow{\begin{pmatrix} -\psi \\ 1 \end{pmatrix}} & T^2 & \xrightarrow{(1 \ \psi)} & T & \xrightarrow{0} & \tau_R(T) \cong \Sigma(T)
\end{array}$$

It follows that τ_R induces an isomorphism from $\underline{\text{Hom}}_A(\Omega T, T)$ to $\underline{\text{Hom}}_A(\tau_R(\Omega T), \tau_R(T))$ as required. By Proposition 2.2, we conclude that τ_R is an equivalence. \square

Suppose M is an indecomposable, non-projective A -module. Then $\tau_R(M)$ is defined by the triangle $M \otimes_R A \rightarrow M \rightarrow \tau_R(M) \rightarrow$, and it follows that we have a short exact sequence $0 \rightarrow \Omega\tau_R(M) \rightarrow M \otimes_R A \rightarrow M \rightarrow 0$. On one hand, the map $M \otimes_R A \rightarrow M$ can be viewed as a relative R -projective cover of M , making $\Omega\tau_R(M)$ a relative syzygy of M . In particular, $\tau_R(k)$ then resembles Alperin's construction of endo-trivial modules for certain 2-groups [1]. Alternatively, when $m = 2$ $M_R \in \text{add}(k_R \oplus R)$, so that the map $M \otimes_R A \rightarrow M$ is a right $\text{add}(T \oplus A)$ -approximation of M . Hence it is often straightforward to compute $\Omega\tau_R(M)$ as the kernel of such an approximation.

It is not difficult to see that a spherical stable twist τ_R is not isomorphic to any power of Ω on $\underline{\text{mod}}\text{-}A$, simply by comparing the dimensions of $\tau_R(k_A)$ and $\Omega^n(k_A)$ for $n \in \mathbb{Z}$ (at least when $d = \dim_k A > 4$). Using fancier language, we consider the stable Grothendieck group $\underline{K}_0(A) \cong \mathbb{Z}/d\mathbb{Z}$. Any auto-equivalence of $\underline{\text{mod}}\text{-}A$ induces an automorphism of $\underline{K}_0(A)$, and the syzygy functor Ω corresponds to multiplication by -1 on $\mathbb{Z}/d\mathbb{Z}$. Since $\tau_R(k_A)$ is defined to be the cosyzygy of the kernel of the natural map $k \otimes_R A \rightarrow k$, we see that the image of $\tau_R(k_A)$ in the stable Grothendieck group is $(m-1)\frac{d}{m} + 1$. Hence τ_R corresponds to multiplication by $(m-1)\frac{d}{m} + 1 \neq \pm 1$ in $\mathbb{Z}/d\mathbb{Z}$. It is a more difficult problem to determine whether or not any higher powers of τ_R coincide with powers of Ω . Two different possibilities are illustrated by the dihedral and semidihedral algebras that we examine in the last section.

6. \mathbb{P}^n STABLE TWISTS

We now define additional auto-equivalences of $\underline{\text{mod}}\text{-}A$ that are analogous to the \mathbb{P}^n twists introduced by Huybrechts and Thomas [13]. We continue the notation introduced in the last section. Namely A is an elementary, local symmetric k -algebra and R is the unital subalgebra generated by an element $x \in \text{rad} A$ with $x^m = 0$ and $x^{m-1} \neq 0$ for some $m \geq 2$. We also assume that ${}_R A$ and A_R are both free, and we set $T = k \otimes_R A$.

Definition 6.1. *Let R and A be as above and suppose $y \in A$ commutes with x . Let $H : A \otimes_R A \rightarrow A \otimes_R A$ be the (A, A) -bimodule map sending $1 \otimes 1$ to $y \otimes 1 - 1 \otimes y$ and $\mu : A \otimes_R A \rightarrow A$ the map induced by multiplication. Let C_H be the cone of H in $\underline{\text{lrp}}(A, A)$, and observe that μ factors through C_H since $\mu H = 0$. Let Q be the*

cone of the induced map from C_H to A in $\underline{\text{lrp}}(A, A)$.

$$(6.1) \quad \begin{array}{ccccccc} A \otimes_R A & \xrightarrow{H} & A \otimes_R A & \longrightarrow & C_H & \longrightarrow & (A \otimes_R A)[1] \\ & \searrow 0 & \downarrow \mu & & \downarrow & & \\ & & A & \xlongequal{\quad} & A & & \\ & & \downarrow & & \downarrow & & \\ (A \otimes_R A)[1] & \longrightarrow & C_\mu & \longrightarrow & Q & \longrightarrow & (A \otimes_R A)[2] \end{array}$$

We write $\rho_{R,y} : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$ for the endo-functor induced by $- \otimes_A Q$. If $\underline{\text{End}}_A(A/xA) \cong k[\psi]/(\psi^{(n+1)})$ with ψ induced by left multiplication by y , then we call $\rho_{R,y}$ a \mathbb{P}^n **stable twist**.

Theorem 6.2. A \mathbb{P}^n **stable twist** $\rho = \rho_{R,y}$ is an auto-equivalence of $\underline{\text{mod}}\text{-}A$.

The proof is similar to that of Theorem 5.2. In particular, we start by analyzing the action of ρ on the spanning class $\mathcal{C} = \{T\} \cup T^\perp$. In Lemmas 6.3 and 6.4, we show that $\rho : \underline{\text{Hom}}_A(X, Y[i]) \rightarrow \underline{\text{Hom}}_A(\rho(X), \rho(Y[i]))$ is bijective for any $X, Y \in \mathcal{C}$. The theorem then follows easily by Propositions 2.2 and 2.3.

Lemma 6.3. *Restricted to T^\perp , ρ is isomorphic to the identity functor.*

Proof. As in the proof of Theorem 5.2, T^\perp coincides with the full subcategory of $\underline{\text{mod}}\text{-}A$ consisting of A -modules that are free over R . By construction, for $X \in T^\perp$, $X \otimes_A C_H$ is projective. Hence the map $A \rightarrow Q$ induces isomorphisms $X \rightarrow X \otimes_A Q$ in $\underline{\text{mod}}\text{-}A$. \square

Lemma 6.4. *Restricted to $\text{add}(T \oplus T[-1])$, ρ is isomorphic to Σ^2 .*

Proof. Tensoring the triangle $A \otimes_R A \xrightarrow{H} A \otimes_R A \rightarrow C_H \rightarrow$ with T_A and using Lemma 4.1, we obtain the top triangle in the morphism of triangles below.

$$(6.2) \quad \begin{array}{ccccc} T^{n+1} & \xrightarrow{\begin{pmatrix} -y & & & & \\ 1 & -y & & & \\ & \ddots & \ddots & & \\ & & 1 & -y & \end{pmatrix}} & T^{n+1} & \longrightarrow & T \otimes_A C_H \longrightarrow \\ \downarrow \begin{pmatrix} 1 & & & -y^n \\ & \ddots & & \vdots \\ & & 1 & -y \\ & & & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 1 & y & \cdots & y^{n-1} \\ & \ddots & \ddots & & \vdots \\ & & 1 & y & \\ 1 & y & y^2 & \cdots & y^n \end{pmatrix} & & \downarrow \cong \\ T^{n+1} & \xrightarrow{\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}} & T^{n+1} & \xrightarrow{\begin{pmatrix} 0 & \cdots & 1 \\ 0 & & 0 \end{pmatrix}} & T \oplus T[1] \longrightarrow \end{array}$$

As the first two vertical maps are isomorphisms, so is the third. Furthermore, as in the proof of Theorem 5.2, the map $\mu : A \otimes_R A \rightarrow A$ induces the map $1_T \otimes \mu : (1_T \ \psi \ \cdots \ \psi^n) : T^{n+1} \rightarrow T$. Under the above isomorphism of triangles this map corresponds to the projection of T^{n+1} onto its $(n+1)^{\text{th}}$ factor. Consequently, the induced map $T \oplus T[1] \cong T \otimes_A C_H \rightarrow T \otimes_A A \cong T$ is the projection onto T . As $\rho(T)$ is defined as the cone of this map, we see that $\rho(T) \cong T[2]$:

$$(6.3) \quad \begin{array}{ccccccc} \rho(T)[-1] & \longrightarrow & T \otimes_A C_H & \longrightarrow & T & \longrightarrow & \rho(T) \\ \downarrow \cong & & \downarrow \cong & & \parallel & & \downarrow \cong \\ T[1] & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & T \oplus T[1] & \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} & T & \xrightarrow{0} & T[2] \end{array}$$

Of course it follows now that $\rho(T[-1]) \cong T[1]$, but notice that we can also suspend the above argument. We point out that $\Sigma(\psi)$ is also induced by multiplication by y on $T[-1] \cong A/x^{m-1}A \cong xA$.

We now compute the effect of ρ on $\underline{\text{Hom}}_A(T, T) = k[\psi]/(\psi^{n+1})$ and on $\underline{\text{Hom}}_A(\Omega T, T)$. When we tensor the map $\psi : T \rightarrow T$ with the triangle $A \otimes_R A \rightarrow A \otimes_R A \rightarrow C_H$ and pass to the isomorphic triangle in (6.2), we see that the resulting endomorphism of the left-most T^{n+1} is

$$\begin{pmatrix} 1 & & & -y^n \\ & \ddots & & \vdots \\ & & 1 & -y \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & -y^n \\ & \ddots & & \vdots \\ & & 1 & -y \\ & & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \cdots & -y^n & 0 \\ 1 & 0 & \cdots & -y^{n-1} & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & y \end{pmatrix}.$$

Restricted to the $(n+1)^{th}$ factor of T we thus obtain the endomorphism ψ , and hence we have $\Sigma(\psi)$ for the $T[1]$ -component of the induced endomorphism of $T \otimes_A C_H \cong T \oplus T[1]$. Finally, this yields $\Sigma^2(\psi)$ for the corresponding endomorphism of $\rho(T) \cong T[2]$.

We now compute the effect of ρ on the map $\xi \in \underline{\text{Hom}}_A(\Omega T, T)$. By Lemma 4.2, $FG(\xi) : (\Omega T)^{n+1} \rightarrow T^{n+1}$ will be given by a diagonal matrix with ξ along the diagonal. This representation of $FG(\xi)$ does not change when we pass to the isomorphic triangles as in (6.2), and hence we see that the induced map $\Omega(T) \oplus \Omega(T)[1] \rightarrow T \oplus T[1]$ is $\begin{pmatrix} \xi & 0 \\ 0 & \Sigma\xi \end{pmatrix}$. For the second cone construction, we use triangles as in (6.3) and its de-suspension to compute $\rho(\xi) = \Sigma^2(\xi)$:

$$(6.4) \quad \begin{array}{ccccccc} \Omega(T)[1] & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & \Omega T \oplus \Omega T[1] & \xrightarrow{(1 \ 0)} & \Omega T & \xrightarrow{0} & \Omega T[2] \cong \rho(\Omega T) \\ \downarrow \Sigma\xi & & \downarrow \begin{pmatrix} \xi & 0 \\ 0 & \Sigma\xi \end{pmatrix} & & \downarrow \xi & & \downarrow \Sigma^2(\xi) \\ T[1] & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & T \oplus T[1] & \xrightarrow{(1 \ 0)} & T & \xrightarrow{0} & T[2] \cong \rho(T) \end{array}$$

Finally, since all maps between T and its suspensions are generated by ψ and ξ , on which ρ acts as Σ^2 , we conclude that the restriction of ρ to $\text{add}(T \oplus T[-1])$ is isomorphic to Σ^2 . \square

Proposition 6.5. *Let $\rho = \rho_{R,y}$ be a \mathbb{P}^n stable twist. Then*

$$\rho_{R,y}(k) \cong \Omega^{-2}(\Omega_R^2(k)),$$

where $\Omega_R(M)$ denotes the kernel of a relatively R -projective cover of M .

Proof. Tensoring the triangles in the diagram (6.1) with k yields the following commutative diagram in which the rows and columns are triangles

$$(6.5) \quad \begin{array}{ccccccc} T & \longrightarrow & \Omega_R(k) & \longrightarrow & \rho_R(k)[-1] & \longrightarrow & T[1] \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ T & \xrightarrow{\psi} & T & \longrightarrow & k \otimes_A C_H & \longrightarrow & T[1] \\ & \searrow 0 & \downarrow & & \downarrow & & \\ & & k & \xlongequal{\quad} & k & & \\ & & \downarrow & & \downarrow & & \\ T[1] & \longrightarrow & \Omega_R(k)[1] & \longrightarrow & \rho_R(k) & \longrightarrow & T[2] \end{array}$$

We claim that the map $T \rightarrow \Omega_R(k)$ in the top left is a relatively R -projective cover. In fact, when we restrict this map to $\underline{\text{mod}}\text{-}R$, we get a split epimorphism $k^{n+1} \rightarrow k^n$ by Lemma 4.1. It follows that for any R -module M , the induced map $\underline{\text{Hom}}_R(M, T_R) \rightarrow \underline{\text{Hom}}_R(M, (\Omega_R(k))_R)$ is onto, and hence $\underline{\text{Hom}}_A(FM, T) \rightarrow \underline{\text{Hom}}_A(FM, \Omega_R(k))$ is also onto; i.e., the map $T \rightarrow \Omega_R(k)$ is a right $F(\underline{\text{mod}}\text{-}R)$ -approximation. It follows that $\Omega_R^2(k) \cong \rho_R(k)[-2]$ as desired. \square

In the context of the derived category of coherent sheaves of a smooth projective variety, Huybrechts and Thomas have shown that a \mathbb{P}^1 -twist coincides with the square of a spherical twist [13]. The analogous statement holds in our context, at least up to a Morita auto-equivalence, whenever $m = 2$ and a \mathbb{P}^1 -twist is defined.

Proposition 6.6. *Let A be a split, local, symmetric k -algebra which is free on either side over the subalgebra $R = k[x]$ where $x^2 = 0$. Suppose that $\underline{\text{End}}_A(k \otimes_R A) \cong k[\psi]/(\psi^2)$ with ψ corresponding to left multiplication by some $y \in A$ that commutes with x . Then $\rho_{R,y}(k) \cong \tau_R^2(k)$. In particular, there is a Morita auto-equivalence F of $\text{mod } A$ (i.e., $F = - \otimes_A {}_1 A_\sigma$ for a twisted bimodule ${}_1 A_\sigma$ with $\sigma \in \text{Out}(A)$) such that $\rho_{R,y} \cong \tau_R^2 \circ F$ as functors on $\text{mod-}A$.*

Proof. Notice we may replace the $\Omega_R(k)[1]$ entry in diagram (6.5) with $\tau_R(k)$. Now $\underline{\text{Hom}}_A(T[1], \tau_R(k)) \cong \underline{\text{Hom}}_A(\tau_R(T), \tau_R(k)) \cong \underline{\text{Hom}}_A(T, k)$ is one-dimensional and $T[1] \cong T$. Thus the map $T[1] \rightarrow \tau_R(k)$ in the bottom row of the diagram is either 0 or else a right $\text{add}(T)$ -approximation. It must be nonzero since $\Sigma\psi$ factors through it. Thus we have $\rho_R(k) \cong \tau_R^2(k)$. The final statement now follows from Linckelmann's theorem. \square

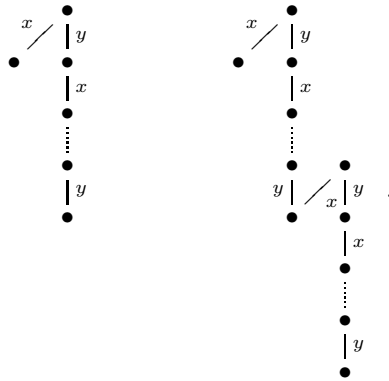
7. EXAMPLES

7.1. Dihedral groups and algebras. Let k be any field and set $A = k\langle x, y \rangle / (x^2, y^2, (xy)^q - (yx)^q)$ for some $q \geq 2$. In general, A is a local, symmetric, special biserial k -algebra of dimension $4q$ with k -basis

$$\{(xy)^i, (xy)^i x, (yx)^{i+1}, (yx)^i y \mid 0 \leq i < q\}.$$

For arbitrary fields k and integers $q \geq 2$, the indecomposable A -modules have been completely described by Ringel [19] and the Auslander-Reiten quiver can be worked out as in [10], Lemma II.7.6 (see also [3], Sections 4.11 and 4.17). If $\text{char}(k) = 2$ and q is a power of 2, then A is isomorphic to the group algebra over k of the dihedral group of order $4q$.

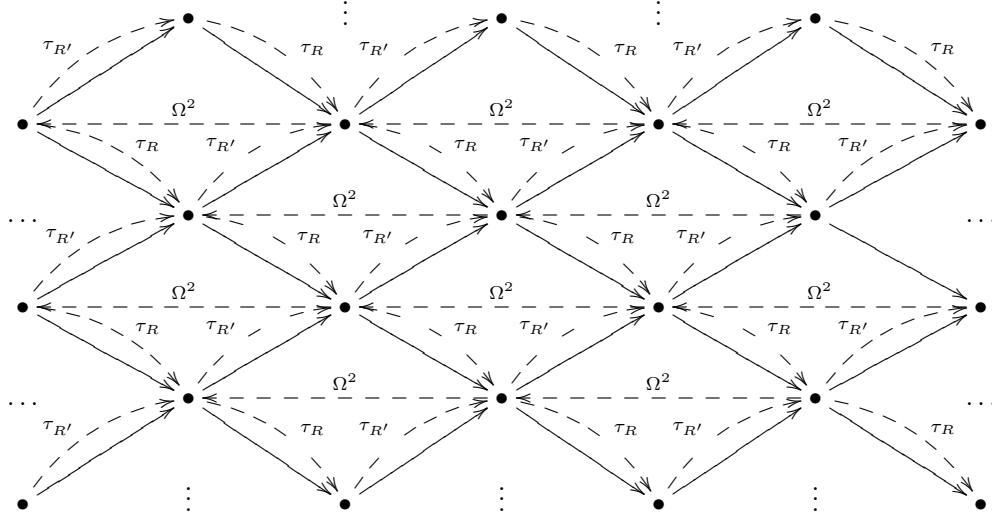
We set $R = k[x] \subset A$, and it is clear that ${}_R A$ and A_R are free R -modules of rank $2q$ with bases given by $\{(yx)^i y^j \mid 0 \leq i < q, j = 0, 1\}$ and $\{y^j (xy)^i \mid 0 \leq i < q, j = 0, 1\}$, respectively. Furthermore, as an (R, R) -bimodule, A decomposes as two copies of the regular bimodule R , generated by 1 and $(yx)^{q-1}y$, and $q-1$ copies of the projective bimodule $R \otimes_k R$, generated by $(yx)^i y$ for $0 \leq i < q-1$; i.e., ${}_R A_R \cong R^2 \oplus (R \otimes_k R)^{q-1}$. Following the notation of Section 4, we see that $T = k \otimes_R A \cong A/xA$ is a uniserial module of dimension $2q$, corresponding to the word $(yx)^{q-1}y$. Clearly, $T_R = k \otimes_R A_R \cong k \otimes_R (R^2 \oplus (R \otimes_k R)^{q-1}) \cong k^2 \oplus R^{q-1}$. Thus the hypotheses of Definition 5.1 are satisfied, and we see that the spherical twist τ_R gives an auto-equivalence of $\text{mod-}A$. Applying τ_R to the simple A -module k , we see that $\tau_R(k)$ and $\tau_R^2(k)$ are the string modules corresponding to the words $x(xy)^{1-q}y^{-1}$ and $x(xy)^{1-q}y^{-1}x(xy)^{1-q}y^{-1}$, respectively, which have graphs



When $A = kD_{4q}$, the endo-trivial module $\tau_R(k)$ coincides with $\Omega^{-1}(L)$ for the endo-trivial module L described in [7], Section 5. It is shown there that the group $T(D_{4q})$ of endo-trivial modules is isomorphic to \mathbb{Z}^2 , generated by the classes of $\Omega(k)$ and L . In this case, the fact that no power of L coincides with a power of $\Omega(k)$ —or, equivalently, that no power of τ_R coincides with a power of Ω —can be seen by restricting to

two different elementary abelian 2-subgroups of rank 2. In the stable category of one such subgroup, $\tau_R(k)$ restricts to k , while in the other it restricts to $\Omega^{-2}(k)$.

Similar arguments do not appear to work in general for showing that no nonzero power of τ_R can coincide with a power of Ω on $\text{mod-}A$. However, we can see that this is indeed still true by considering the action of τ_R on the stable AR-quiver of A . We focus on the two components of the stable AR-quiver \mathcal{C}_k and \mathcal{C}_A containing the simple A -module and $\text{rad}A$, respectively. Each of these components has the shape $\mathbb{Z}A_\infty^\infty$, and Ω induces isomorphisms of translation quivers $\mathcal{C}_k \rightarrow \mathcal{C}_A$ and $\mathcal{C}_A \rightarrow \mathcal{C}_k$ with $\Omega^2 = \tau$ since A is symmetric. It can be seen that $\tau_R(k)$ is one of the two immediate successors of k in \mathcal{C}_k (the other successor is in fact $\tau_{R'}(k)$ for $R' = k[y] \subseteq A$). Furthermore, one checks that $\tau_R^2(k) \not\cong \Omega^{-2}(k) = \tau^{-1}(k)$. Since the stable equivalence τ_R commutes with Ω on objects, it follows that τ_R induces an automorphism of \mathcal{C}_k that can be described as shifting down and to the right one arrow. Again since τ_R commutes with Ω , τ_R induces an automorphism of \mathcal{C}_A of the same form. We sketch a portion of the component \mathcal{C}_k below.



In particular, we see that no nonzero power of τ_R can be isomorphic to a power of Ω . Moreover, in the case where $A = kD_{4q}$ (with q a power of 2), we see that these two components of the stable AR-quiver consist precisely of all the endo-trivial A -modules. This appears interesting, but probably not indicative of what happens more generally.

7.2. Semidihedral groups and algebras. Let k be any field and set

$$A = k\langle x, y \rangle / (x^2, y^4, y^2 - (xy)^{q-1}x - \delta(yx)^q, (xy)^q - (yx)^q)$$

for some $q \geq 2$ and $\delta \in k$. In general, A is a local, symmetric k -algebra of dimension $4q$ with k -basis

$$\{(xy)^i, (xy)^i x, (yx)^{i+1}, (yx)^i y \mid 0 \leq i < q\}.$$

If $\text{char}(k) = 2$, $q = 2^n$ for $n \geq 2$ and $\delta = 1$, then A is isomorphic to the group algebra over k of the semidihedral group of order $4q$.

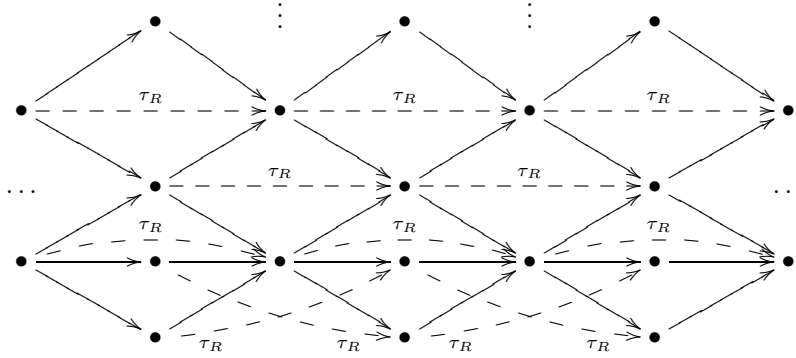
As for the dihedral algebras above, we may set $R = k[x] \subset A$, and it is clear that ${}_R A$ and A_R are free R -modules of rank $2q$ with bases given by $\{(yx)^i y^j \mid 0 \leq i < q, j = 0, 1\}$ and $\{y^j (xy)^i \mid 0 \leq i < q, j = 0, 1\}$, respectively. Furthermore, as an (R, R) -bimodule, A decomposes as two copies of the regular bimodule R , generated by 1 and $(yx)^{q-1}y$, and $q-1$ copies of the projective bimodule $R \otimes_k R$, generated by $(yx)^i y$ for $0 \leq i < q-1$; i.e., ${}_R A_R \cong R^2 \oplus (R \otimes_k R)^{q-1}$. Following the notation of Section 4, we see that $T = k \otimes_R A \cong A/xA$ is a uniserial module of dimension $2q$, corresponding to the word $(yx)^{q-1}y$. Clearly, $T_R = k \otimes_R A_R \cong k \otimes_R (R^2 \oplus (R \otimes_k R)^{q-1}) \cong k^2 \oplus R^{q-1}$. Thus the hypotheses of Definition 5.1 are satisfied, and we see that the spherical twist τ_R gives an auto-equivalence of $\text{mod-}A$. Applying τ_R to the simple A -module k , we see that $\tau_R(k)$ is the string module corresponding to the word $x(xy)^{1-q}y^{-1}$ as in the dihedral case.

When $A = kSD_{4q}$, the endo-trivial module $\tau_R(k)$ coincides with $\Omega^{-1}(L)$ for the endo-trivial module L described in [7], Section 7. It is shown there that the group $T(SD_{4q})$ of endo-trivial modules is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, generated by the classes of $\Omega(k)$ and $\Omega(L)$, which has order 2 (see Theorem 7.1 of [7]). Hence, in this case we have $\tau_R^2(k) \cong \Omega^{-4}(k)$ and it follows that $(-\otimes_k \tau_R(k))^2 \cong \Omega^{-4}$ on $\underline{\text{mod}}\text{-}A$.

It is not clear whether the analogous isomorphism $\tau_R^2 \cong \Omega^{-4}$ holds for all local semidihedral algebras. However, we can again consider the automorphism of the stable AR-quiver of A induced by τ_R . In II.10 of [10], Erdmann shows that the stable AR-quiver of A consists of (infinitely many) $\mathbb{Z}A_\infty$ and $\mathbb{Z}D_\infty$ components, as well as tubes of rank 1 and 2. The simple A -module k lies on the mouth of a $\mathbb{Z}D_\infty$ component. The almost split sequence starting in k can be computed by applying Ω^{-1} to the almost-split sequences starting in $\text{rad}A$, which has the form $0 \rightarrow \text{rad}A \rightarrow A \oplus \text{rad}A/\text{soc}A \rightarrow A/\text{soc}A \rightarrow 0$. It follows that $0 \rightarrow k \rightarrow \Omega^{-1}(\text{rad}A/\text{soc}A) \rightarrow \Omega^{-2}(k) \rightarrow 0$ is an almost-split sequence. Observe that $\text{rad}A/\text{soc}A$ is indecomposable. There is another almost split sequence

$$0 \rightarrow \Omega^2 \tau_R(k) \rightarrow \Omega^{-1}(\text{rad}A/\text{soc}A) \rightarrow \tau_R(k) \rightarrow 0,$$

which is Ω^{-1} applied to the sequence in II.9.5 of [10]. Thus the component of k is preserved by τ_R which has the effect of shifting vertices one unit to the right, except for the two leaves of each sectional D_∞ subtree, which are swapped and then shifted one unit to the right. It is clear that $\tau_R^2(X) \cong \Omega^{-4}(X)$ for each indecomposable A -module X in this component, and for most such X (those not on the mouth), we even have $\tau_R(X) \cong \Omega^{-2}(X)$.



Thus we have $\Omega^4 \tau_R^2(k) \cong k$ for the unique simple A -module k . By Linckelmann's Theorem, we can conclude that $\tau_R^2 \cong \Omega^{-4} \circ F$ on $\underline{\text{mod}}\text{-}A$, where F is a Morita auto-equivalence of $\text{mod}\text{-}A$. Moreover F induces the identity automorphism on this component of the stable AR-quiver.

7.3. Extraspecial p -groups. The construction of the \mathbb{P}^n stable twists in the last section was partially motivated by Alperin's construction of endo-trivial modules using relative syzygies [1], as described in Theorem 3.1 of [8]. In particular, for an odd prime p , we focus on an extraspecial p -group G of order p^3 and exponent p , and demonstrate that \mathbb{P}^{p-1} stable twists are defined for kG (with k an algebraically closed field of characteristic p), and that the endo-trivial kG -modules are recovered as the images of the trivial module k (and its syzygies) under these twists.

Let $G = \langle g, h, z \mid g^p = h^p = z^p = 1, z = [g, h], gz = zg, hz = zh \rangle$ be the extraspecial p -group of order p^3 and exponent p (we follow the notation of [6], §6). Let $H_i = \langle g^i h \rangle$ and $E_i = \langle z, g^i h \rangle = C_G(H_i)$ for $1 \leq i \leq p$, and set $H_{p+1} = \langle g \rangle$ and $E_{p+1} = \langle z, g \rangle = C_G(H_{p+1})$. We also let $A = kG$ and $R_i = kH_i$ for each $1 \leq i \leq p+1$. Notice that $x_i := 1 - g^i h$ (resp. $x_{p+1} = 1 - g$) generates kH_i as an algebra and we make the identification $kH_i = k[x_i]/(x_i^p)$. Furthermore $y = 1 - z \in kG$ commutes with each x_i . We set $T_i = A/x_i A = k_{H_i} \uparrow^G$. In order to show that $\rho = \rho_{R_i, y}$ actually defines a \mathbb{P}^{p-1} stable twist, it suffices to verify $\underline{\text{End}}_{kG}(T_i) \cong k[\psi]/(\psi^p)$ with ψ corresponding to multiplication by y .

To simplify notation, we fix $i = p$ so that we can work with $H = H_p$ and $R = R_p$. (In fact, as G has automorphisms sending $g \mapsto g, h \mapsto g^i h, z \mapsto z$ or sending $g \mapsto h, h \mapsto g, z \mapsto z^{-1}$, the other cases can be deduced from our calculations here. Moreover, it will follow that all the $\rho_{R_i, y}$ differ from one another by

Morita equivalences.) With respect to the subgroup H , G has the double coset decomposition

$$G = \bigcup_{j=0}^{p-1} H z^j H \cup \bigcup_{l=0}^{p-1} H h^l H,$$

with $H z^j H = H z^j$ as z is central. It follows that as a (kH, kH) -bimodule

$$kG = \bigoplus_{j=0}^{p-1} k(H z^j H) \oplus \bigoplus_{l=0}^{p-1} k(H h^l H) \cong \bigoplus_{j=0}^{p-1} (kH \otimes z^j) \oplus \bigoplus_{l=0}^{p-1} (kH \otimes h^l) \otimes_k kH,$$

as $H \cap H z^j = H$ for each j and $H \cap H h^l = 1$ for each l . Therefore, by Mackey's theorem we have

$$k_H \uparrow^G \downarrow_H \cong \bigoplus_{j=0}^{p-1} (k_H \otimes z^j) \oplus \bigoplus_{l=0}^{p-1} (k \otimes h^l) \uparrow^H \cong k_H^p \oplus kH^p,$$

where $k_H \otimes z^j \cong k_H$ for each j and $(k \otimes h^l) \uparrow^H \cong kH$ for each l . In particular, for $T = k_H \uparrow^G$, we see that $\underline{\text{Hom}}_{kG}(T, T) \cong \underline{\text{Hom}}_{kH}(k_H, T \downarrow_H)$ is p -dimensional. Moreover, the automorphism of T induced by multiplication by z restricts to an automorphism of the non-projective part of $T \downarrow_H$, cyclicly permuting the p components of the form $k \otimes z^j$. Let ψ be the endomorphism of T induced by multiplication by $y = 1 - z$. Clearly $\psi^p = 0$ and the restriction of ψ^j to an endomorphism of the non-projective part of $T \downarrow_H$ is non-zero for each $j < p$. In fact, this restriction is non-zero in the stable category as well since it is an endomorphism of a (non-projective) semisimple module. It follows that the non-zero powers of ψ are linearly independent in $\underline{\text{End}}_{kH}(T \downarrow_H)$ and hence in $\underline{\text{End}}_{kG}(T)$ as well. Hence we conclude that $\underline{\text{End}}_{kG}(T) = k[\psi]/(\psi^p)$ as required.

Finally, Proposition 6.5 shows that the image of the trivial module k_G under $\rho_{R_i, y}$ coincides with the endo-trivial module N_i described in Theorem 3.1(c) of [8].

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